DIFFERENTIAL EQUATIONS

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An equation involving independent variables, dependent variables and their derivatives/differentials is called a **differential equation**.

Differential equation with one independent variable is called **ordinary differential equation** and more than one variable is termed as **partial differential equation**

ORDER & DEGREE OF DIFFERENTIAL EQUATION

Order of differential equation is the order of the derivative of the highest order.

Degree is the degree of the highest order coefficient provided the differential coefficients are made free from radicals and fractions.

LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation is a linear differential equation if it is expressible in the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where P_0, P_1, P_2, ……… P_{n-1} P_n and Q are either constants or function of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be **linear** differential equation. Otherwise, it is a **non-linear** differential equation.

It follows from the above definition that a differential equation will be **non-linear differential equation if–**

1. Its degree is more than one.
2. Any of the differential coefficient has exponent more than one.
3. Exponent of the dependent variable is more than one.
4. Products containing dependent variable and its differential coefficients are present.

The following are some of the examples of differential equations:

1. $\frac{dy}{dx} = x \log x$ *(order = 1, degree = 1)*

2. dy = *cos* x dx *(order = 1, degree =1)*

3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = x^4$ *(order =2, degree = 1)*

4. $\left[1 + \frac{dy}{dx}\right]^4 = 6\left(\frac{d^2y}{dx^2}\right)^2$ *(order =2, degree = 2)*

SOLUTION OF DIFFERENTIAL EQUATION

The general solution of the equation $\frac{dy}{dx} = f(x)$ is $y = \int f(x)\,dx + C$.

(I) VARIABLE SEPARATION METHOD

The general solution of the equation $\frac{dy}{dx} = f(x)\,g(y)$ is given by $\int \frac{1}{g(y)}dy = \int f(x)\,dx + C$.

EXAMPLE:

1. Solve $y' = \sin x \cos y$

Sol. $\frac{dy}{dx} == \sin x \cos y \Rightarrow \frac{dy}{\cos y} = \sin x\,dx$

$\Rightarrow \int \sec y\,dy = \int \sin x\,dx = \log|\sec y + \tan y| = -\cos x + C$

2. Solve: $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$.

Sol. The given equation can be written as $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$,

Integrating, we get $\int \frac{1}{1+x^2}dx + \int \frac{1}{1+y^2}dy = C$,

where C is an arbitrary constant

or $\tan^{-1} x + \tan^{-1} y = C$

or $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = C$

or $\frac{x+y}{1-xy} = \tan C = K$ (say)

or $\frac{x+y}{1-xy} = K$, which is the required solution.

(II) SUBSTITUTION METHOD

To solve an equation of the type $\frac{dy}{dx} = f(ax + by + c)$,

Substitute $z = ax + by + c \Rightarrow \frac{dz}{dx} = a + b\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dx} - a}{b}$

$\therefore \frac{dy}{dx} = f(ax + by + c) \Rightarrow \frac{\frac{dz}{dx} - a}{b} = f(z) \Rightarrow \frac{dz}{dx} = bf(z) + a \Rightarrow \frac{dz}{bf(z)+a} = dx$

$\therefore \Rightarrow \int \frac{dz}{bf(z)+a} = x + C$, *where* $z = ax + by + c$ *(variables separation method)*

EXAMPLE:

3. Solve $(x + y)^2 \frac{dy}{dx} = a^2$

Sol. Put x + y = v, so that $1 + \frac{dy}{dx} = \frac{dv}{dx}$ or $\frac{dy}{dx} = \frac{dv}{dx} - 1$.

Substituting above values in the given differential equation, we have

$$v^2 \left(\frac{dv}{dx} - 1\right) = a^2$$

or $\frac{dv}{dx} = \frac{a^2}{v^2} + 1 = \frac{a^2 + v^2}{v^2}$

or $\frac{v^2}{v^2 + a^2} dv = dx$ or $\frac{v^2 + a^2 - a^2}{v^2 + a^2} dv = dx$

or $\left(1 - \frac{a^2}{v^2 + a^2}\right) dv = dx$.

On integration, we get $v - a \tan^{-1} \frac{v}{a} = x + C$

or $(x + y) - a \tan^{-1}\left(\frac{x + y}{a}\right) = x + C$ which is the required solution.

(III) HOMOGENEOUS EQUATIONS

An equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}, g(x, y) \neq 0 \quad \forall \ x, y$ is called a homogeneous equation,

if (x, y) and g (x, y) are homogeneous functions of same degree in x and y.

Let $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$ and $g(x, y) = x^n \psi\left(\frac{y}{x}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ reduces to $\frac{dy}{dy} = \frac{x^n \phi(y/x)}{x^n \psi(y/x)} = \frac{\phi(y/x)}{\psi(y/x)}$

Let us call $\frac{\varphi(y/x)}{\psi(y/x)}$ as $F(y/x)$.

$\therefore$ The above differential equation can be expressed as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \dots (1)$$

Let y = vx $\therefore \frac{dy}{dx} = v.1 + x \frac{dv}{dx}$

$\therefore (1) \Rightarrow v + x \frac{dv}{dx} = F(v)$

$\Rightarrow x \frac{dv}{dx} = F(v) - v \Rightarrow \frac{dv}{F(v) - v} = \frac{dx}{x}$

$\Rightarrow \int \frac{dv}{F(v) - v} = \log x + c$

$\therefore$ The given homogeneous equation is solved.

EXAMPLES:

4. Solve $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$

Sol. The given equation is

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \text{ or } x\frac{dy}{dx} = y + \sqrt{x^2 + y^2} \text{ or } \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \dots (1)$$

This is a homogeneous equation.

Let y = vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore (1) \Rightarrow v + x\frac{dy}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$\Rightarrow \quad x\frac{dv}{dx} = (v + \sqrt{1 + v^2}) - v = \sqrt{1 + v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x} \Rightarrow \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} + \log C$$

$$\Rightarrow \quad \log\ (v + \sqrt{1 + v^2}) = \log x + \log C \qquad \text{(Use } v = \tan\theta\text{)}$$

$$\Rightarrow \quad \log\left(\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right) = \log Cx$$

$$\Rightarrow \quad \frac{y + \sqrt{x^2 + y^2}}{x} = Cx \Rightarrow y + \sqrt{x^2 + y^2} = Cx^2$$

5. Solve $2xyy' = x^2 + 3y^2$

Sol. The given equation is $2xy\frac{dy}{dx} = x^2 + 3y^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \quad \dots (1)$$

This is a homogenous equation.

Let $y = vx \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore (1) \Rightarrow \ v + x\frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{2x.vx}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v^2}{2v} \quad \Rightarrow \quad \frac{2vdv}{1 + v^2} = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{2v}{1 + v^2} dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow \quad \log\ (1 + v^2) = \log x + \log\ C \Rightarrow \log\left(1 + \frac{y^2}{x^2}\right) = \log Cx$$

$$\Rightarrow \quad \frac{x^2 + y^2}{x^2} = Cx \quad \Rightarrow \quad x^2 + y^2 = Cx^{3.}$$

6. Solve (x + y) (dx – dy) = dx + dy

Sol. The given equation can be written as $\frac{dy}{dx} = \frac{x+y-1}{x+y+1}$

Putting x + y = v , so that $\frac{dy}{dx} = \frac{dv}{dx} - 1$, we get

$\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$ or $\frac{dv}{dx} = \frac{2v}{v+1}$ or $\frac{v+1}{2v}$ dv = dx or $\left(\frac{1}{2} + \frac{1}{2v}\right)$ dv = dx

Integrating, we get

v + log v = 2x + c.

Therefore the required solution will be (x + y) + log (x + y) = 2x + c or y – x + log (x + y) = c

(IV) EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

If a first order, first degree differential equation is of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \ldots \text{(i)}$$

Then, it can be reduced to homogeneous form by certain substitutions, as explained below;

Put x = X + h, = Y + k ; where h and k are constants, which are to be determined.

$\frac{dy}{dx} = \frac{dY}{dX}$. Substituting these values in (i), we have,

$$\frac{dy}{dx} = \frac{(a_1X + b_1Y) + a_1h + b_1k + c_1}{(a_2X + b_2Y) + a_2h + b_2k + c_2} \quad \ldots \text{(ii)}$$

Now h, k will be chosen such that $a_1h + b_1k + c_1 = 0$ … (iii) and $a_2h + b_2k + c_2 = 0$

i.e. $\frac{h}{b_1c_2 - b_2c_1} = \frac{k}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$ … (iv)

For these values of h and k the equation (ii) reduces to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

which is a homogeneous differential equation and can be solved by the substitution y = vx.
Replacing X and Y in the solution so obtained by x – h and y – k respectively, we can obtain the required solution in terms of x and y.

(V) **<u>CASE 1</u> :**

The general solution of the differential equation $\frac{dy}{dx}$ + Py = Q; Q being function of x only, is given by

$$ye^{\int Pdx} = \int \left(Qe^{\int Pdx}\right) dx + C.$$

or y = $e^{-\int Pdx}\left(\int Q\,(\text{I.F})dx + C\right)$

where I.F. is known as **integrating factor** $= e^{\int Pdx}$

CASE 2 :

The general solution of the differential equation $\frac{dx}{dy}$ + Px = Q; Q being function of y only, is given by

$$xe^{\int Pdy} = \int \left(Qe^{\int Pdy}\right) dy + C.$$

(VI) BERNOULI'S EQUATION

$\frac{dy}{dx}$ + Py = Qy^n; where P and Q being function of x only

The given differential equation can be written as $y^{-n}\frac{dy}{dx} + P.y^{1-n} = Q$

Put $y^{1-n} = z$ $\Rightarrow (1-n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}.$

Now (i) reduces to $\frac{dz}{dx} + (1-n)Pz = (1-n)Q$ which is a linear differential in z.

$\therefore$ I.F. = $e^{\int (1-n)P\,dx}$ So solution is

$$z.e^{\int (1-n)P\,dx} = (1-n)\int Q\, e^{\int (1-n)P\,dx} dx + C \text{ where } z = y^{1-n}$$

(VII) To solve an equation of the type $f'(y)\frac{dy}{dx} + P\, f(y) = Q$; P and Q being functions of x only,

Substitute $f(y) = z$

EXAMPLES:

7. Solve: y' – 2y = cos 3x.

Sol. The given differential equation is $\frac{dy}{dx} - 2y = \cos 3x$

This is linear differential equation of first order.

Hence P = –2, Q = cos 3x. I.F. = $e^{Px} = e^{-2x}$

$\therefore (1) \Rightarrow e^{-2x}\left[\frac{dy}{dx} - 2y\right] - e^{-2x}\, 2\cos 3x$

$\Rightarrow e^{-2x}\frac{dy}{dx} + y\frac{d}{dx}(e^{-2x}) = e^{-2x}\cos 3x$

$\Rightarrow \frac{d}{dx}(ye^{-2x}) = e^{-2x}\cos 3x$

$\Rightarrow ye^{-2z} = \int e^{-2x}\cos 3x\, dx + Ce^{2x}$

8. $\frac{dy}{dx} = x^3y^3 - xy$

Sol. The given equation is $\frac{dy}{dx} + xy = x^3y^3$

Dividing by y^3, we get

$$y^{-3}\frac{dy}{dx}+\frac{x}{y^2}=x^3 \quad \dots (1)$$

Put $y^{-2}=z \;\Rightarrow\; -2y^{-3}\frac{dy}{dx}=\frac{dz}{dx} \qquad \Rightarrow\; y^{-3}\frac{dy}{dx}=-\frac{1}{2}\frac{dz}{dx}$

Now (1) becomes

$$-\frac{1}{2}\frac{dz}{dx}+xz=x^2 \Rightarrow \frac{dz}{dx}-2xz=-2x^3$$, which is linear in z.

So, I.F. $e^{-\int 2x\,dx}=e^{-x^2}$

$\therefore$ The solution is $ze^{-x^2}=-z\int e^{-x^2}x^3\ dx+C$

Put $-x^2=t \Rightarrow\; -2x\,dx=dt$

$$\frac{1}{y}e^{-x^2}=-\int te^{t\,dt}+C=-e^t(t-1)+C=e^{-x^2}(x^2+1)+C.$$